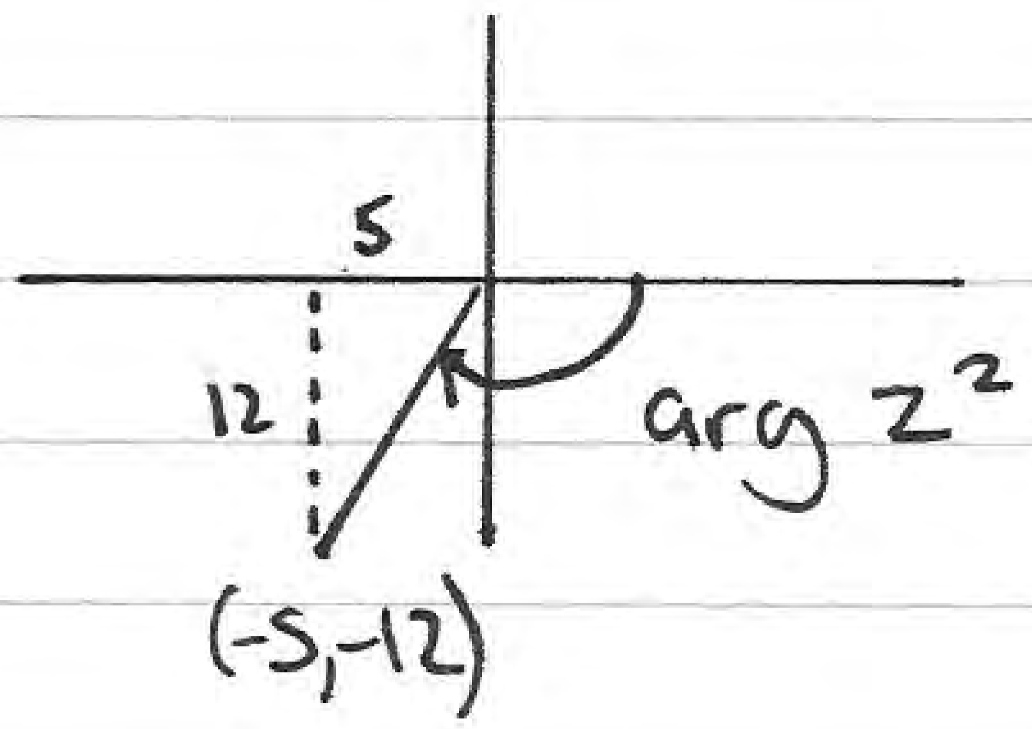
FP1 June 2010

1) $z^2 = (2-3i)(2-3i) = 4-6i-6i+9i^2 = (4-9)-12i$

$= -5-12i$ #

b) $|z^2| = \sqrt{5^2+12^2} = \underline{13}$

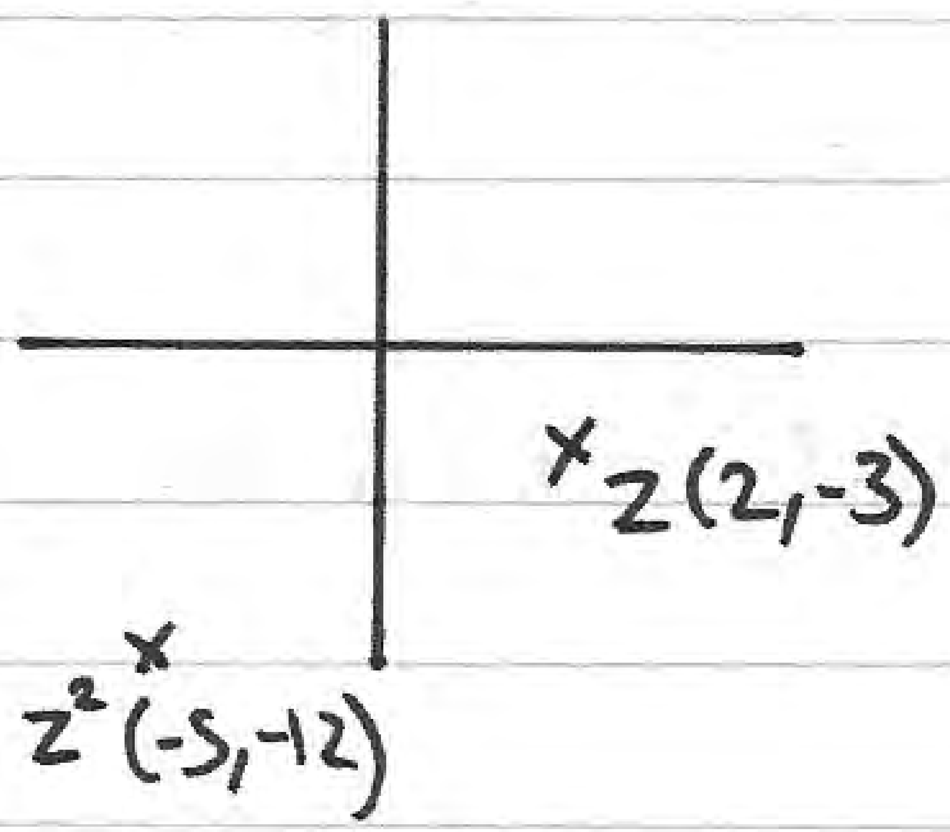
c)



$\arg(z^2) = -\left(\pi - \tan^{-1}\left(\frac{12}{5}\right)\right)$

$\arg(z^2) = \underline{-1.97^c}$

d)



2) $M = \begin{pmatrix} 2a & 3 \\ 6 & a \end{pmatrix}$ $a=2 \Rightarrow M = \begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix}$

$\det M = 4\times2 - 3\times6 = -10 \Rightarrow M^{-1} = \frac{-1}{10}\begin{pmatrix} 2 & -3 \\ -6 & 4 \end{pmatrix}$

b) Singular if $\det M = 0 \Rightarrow 2a^2 = 18 \Rightarrow a^2 = 9 \Rightarrow \underline{a = \pm 3}$

3) $f(x) = x^3 - \frac{7}{x} + 2$ $\quad f(1.4) = -0.256 < 0$

$f(1.5) = +0.708 > 0$

change of sign $\Rightarrow \alpha \in [1.4, 1.5]$

b)

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$f\left(\frac{a+b}{2}\right)$
1.4	-0.26	1.5	0.71	1.45	0.22
1.4	-0.26	1.45	0.22	1.425	-0.02
1.425	-0.02	1.45	0.22		

$\Rightarrow \alpha \in [1.425, 1.45]$

c) $f'(x) = 3x^2 + 7x^{-2}$

$x_0 = 1.45 \qquad x_1 = 1.45 - \dfrac{f(1.45)}{f'(1.45)} = \underline{1.427}$

4) $f(x) = (x+3)(x^2+ax+b) = x^3+x^2+44x+150$

$3b = 150 \Rightarrow b = 50$

$3x^2 + ax^2 = 1x^2 \Rightarrow 3+a \equiv 1 \Rightarrow a = -2$

b) $f(x) = (x+3)(x^2-2x+50)$

$x^2-2x+50=0 \Rightarrow (x-1)^2-1+50=0$

$\Rightarrow (x-1)^2 = -49$

$\Rightarrow x-1 = \pm 7i$

$\Rightarrow x = 1 \pm 7i$

$\alpha = -3,\ 1+7i,\ 1-7i$

c) $-3+1+7i+1-7i = \underline{-1}$

5) $y^2 = 20x = 4ax \Rightarrow a = 5$ focus $(5,0)$ directrix $x+5=$

a) $P(5t^2, 10t)$ $y^2 = 20(5t^2) = 100t^2 \Rightarrow y = \sqrt{100t^2} = 10t$ #

b) $A(80,40)$ $S(5,0)$ $M_{SA} = \frac{40}{75} = \frac{8}{15}$

6a) $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix}$

d) $AB = \begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} k & 1 \\ c & -6 \end{pmatrix} = \begin{pmatrix} 6k + c & 0 \\ 4k + 2c & -8 \end{pmatrix}$

e) $6k + c = 8 \Rightarrow 6k + c = 8$
$4k + 2c = 0 \Rightarrow 2k + c = 0$ −
$4k = 8 \Rightarrow \underline{k = 2} \Rightarrow \underline{c = -4}$

6) $f(k) = 2^k + 6^k$
$\Rightarrow f(k+1) = 2^{k+1} + 6^{k+1} = 2(2^k) + 6(6^k)$
$= 6(2^k) - 4(2^k) + 6(6^k)$
$= 6(2^k + 6^k) - 4(2^k)$
$= 6f(k) - 4(2^k)$ #

b) $n=1$ $f(1) = 2^1 + 6^1 = 8 = 8 \times 1$ hence $\div$ by 8
$n=2$ $f(2) = 2^2 + 6^2 = 40 = 8 \times 5$ hence $\div$ by 8

$n=k$ $f(k) = 2^k + 6^k$ assume is $\div$ by 8

$n=k+1$ $f(k+1) = 6f(k) - 4(2^k) = 6(8\lambda) - 4 \times 2(2^{k-1})$
$= 8(6\lambda - (2^{k-1}))$
hence $\div$ by 8

true for $n=1$, true for $n=k+1$ if true for $n=k$
$\therefore$ by induction true for all $n \in \mathbb{Z}^+$ #

7) $xy = c^2$ $\quad (ct, \frac{c}{t})$ $\quad ct = 3c \Rightarrow t = 3$

$y = \frac{c}{t} = \frac{c}{3}$

b) $y = c^2x^{-1} \Rightarrow \frac{dy}{dx} = -c^2x^{-2} = \frac{-c^2}{x^2} \Rightarrow m_t = \frac{-\cancel{c^2}}{\cancel{c^2}t^2}$

$m_t = -\frac{1}{t^2} \Rightarrow m_n = t^2$

$y - \frac{c}{3} = t^2(x - ct)$ $\quad t=3 \Rightarrow y - \frac{c}{3} = 9(x - 3c)$

$\times 3 \Rightarrow 3y - c = 27(x - 3c) \Rightarrow 3y - c = 27x - 81c$

$\Rightarrow 3y = 27x - 80c$ #

c) $xy = c^2 \Rightarrow x\left(\frac{27x - 80c}{3}\right) = c^2$

$\Rightarrow 27x^2 - 80cx - 3c^2 = 0$ $\quad x = 3c$ is a root

$\Rightarrow (27x + c)(x - 3c) \Rightarrow x = -\frac{c}{27} \Rightarrow t = -\frac{1}{27}$

$y = -27c$

$B\left(-\frac{c}{27}, -27c\right)$

8) $n=1$ $\quad \sum_1^1 r^2 = 1^2 = 1$ $\quad \frac{1}{6}(1)(2)(3) = 1$ $\therefore$ true for $n=1$

$n=2$ $\quad \sum_1^2 r^2 = 1^2 + 2^2 = 5$ $\quad \frac{1}{6}(2)(3)(5) = 5$ $\therefore$ true for $n=2$

$n=k$ $\quad \sum_1^k r^2 = \frac{1}{6}k(k+1)(2k+1)$ $\quad$ assume to be true

$n=k+1$ $\quad \sum_1^{k+1} r^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$

$$\sum_1^{k+1} r^2 = \sum_1^{k} r^2 + (k+1)^2 = \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

$$= \tfrac{1}{6}(k+1)\left[k(2k+1)+6(k+1)\right]$$

$$= \tfrac{1}{6}(k+1)\left[2k^2+7k+6\right]$$

$$= \tfrac{1}{6}(k+1)(k+2)(2k+3) \quad \text{as required.}$$

$\therefore$ true for $n=1$, true for $n=k+1$ if true for $n=k$ so by induction true for all $n \in \mathbb{Z}$.

b)

$$\sum(r+2)(r+3) = \sum r^2 + 5\sum r + \sum 6$$

$$= \tfrac{1}{6}n(n+1)(2n+1) + 5\left(\tfrac{1}{2}n(n+1)\right) + 6n$$

$$= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{15}{6}n(n+1) + \tfrac{36}{6}n$$

$$= \tfrac{1}{6}n\left[(n+1)(2n+1) + 15(n+1) + 36\right]$$

$$= \tfrac{1}{6}n\left[2n^2+3n+1+15n+15+36\right]$$

$$= \tfrac{1}{6}n\left[2n^2+18n+52\right]$$

$$= \tfrac{1}{3}n\left[n^2+9n+26\right] \quad a=9 \quad b=26.$$

c)

$$\sum_{n+1}^{2n}(r+2)(r+3) = \tfrac{1}{3}(2n)\left[(2n)^2+9(2n)+26\right] - \tfrac{1}{3}n(n^2+9n+26)$$

$$= \tfrac{1}{3}n\left[2(4n^2+18n+26)-(n^2+9n+26)\right]$$

$$= \tfrac{1}{3}n\left[7n^2+27n+26\right] \quad \#$$